Stagnation-Point Flow over an Exponentially Shrinking/Stretching Sheet

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The steady two-dimensional stagnation-point flow of an incompressible viscous fluid over an exponentially shrinking/stretching sheet is studied. The shrinking/stretching velocity, the free stream velocity, and the surface temperature are assumed to vary in a power-law form with the distance from the stagnation point. The governing partial differential equations are transformed into a system of ordinary differential equations before being solved numerically by a finite difference scheme known as the Keller-box method. The features of the flow and heat transfer characteristics for different values of the governing parameters are analyzed and discussed. It is found that dual solutions exist for the shrinking case, while for the stretching case, the solution is unique.

Key words: Stagnation-Point; Exponentially Shrinking; Boundary Layer; Heat Transfer; Dual Solutions.

1. Introduction

Started from the last century, there have been numerous sophisticated studies on boundary layer flow. The effects of viscosity and thermal conductivity are important in this layer. Thus, this leads to an urge to understand the underlying physical, mathematical, and modelling concepts inherent in boundary layers. In reality, the majority of applications in industrial manufacturing processes have to deal with fluid flow and heat transfer behaviours. Examples include the polymer sheet extrusion from a dye, gaseous diffusion, heat pipes, drawing of plastic film, etc. Such processes play an important role to determine the quality of the final products as explained by Karwe and Jaluria [1, 2].

Crane [3] has first initiated to discuss the two-dimensional steady flow of an incompressible viscous fluid induced by a linearly stretching plate. The boundary layer equations were simplified using a similarity transformation, which transformed the governing partial differential equations to a single ordinary differential equation. Since then, there were similar flows that have been considered by several researchers [4 – 8]. Such similar flows have been studied extensively in various aspects, for example dealing with suction/injection, stretching, the magnetohydrodynamic (MHD) effect, radiation or considering the non-Newtonian fluids. Magyari and Keller [9] reported the similarity solutions describing the steady plane (flow and thermal) boundary layers on an exponentially stretching continuous surface with an exponential temperature distribution. This problem was then extended by Bidin and Nazar [10], Sajid and Hayat [11], and Nadeem et al. [12, 13] to include the effect of thermal radiation, while Pal [14] and Ishak [15] studied the similar problem but in the presence of a magnetic field. Sanjayanand and Khan [16] studied the heat and mass transfer in a viscoelastic boundary layer flow over an exponentially stretching sheet. The mixed convection flow of a micropolar fluid over an exponentially stretching sheet was considered by El-Aziz [17]. The problems in non-Newtonian fluids considered in [16, 17] do not admit similarity solutions, and thus the authors reported local similarity solutions with certain assumptions.

Recently, the shrinking aspect has become a brand new topic. The abnormal behaviour in the fluid flow due to a shrinking sheet has gained attention from several researchers. However, the work on it is relatively little. The flow induced by a shrinking sheet was first discussed by Miklavčič and Wang [18], where the existence and (non)uniqueness of solutions in both numerical and exact solutions were proven. In extension to that, Fang [19] has carried out the shrinking problem to power law surface velocity with mass transfer. It was shown that the solution only exists with mass suction for the rapidly shrinking sheet problem. Furthermore, Wang [20] has investigated that the shrinking sheet problem has many unique characteristics. Later on, Sajid et al. [21] concerned with the MHD rotating flow over a shrinking surface. It was found that the results in the case of hydrodynamic flow are not stable for the shrinking surface and only meaningful in the presence of a magnetic field. The flow over a shrinking sheet in a porous medium was studied by Nadeem and Awais [22]. On the other hand, Ishak et al. [23] solved numerically the micropolar fluid flow problem over a linearly shrinking sheet, and found that different from the stretching case, the solutions are not unique. Very recently, Nadeem et al. [24 – 26] studied the stagnation point flow over a shrinking sheet in non-Newtonian fluids.

Motivated by the above investigations, in this paper we study the steady two dimensional stagnation point flow over an exponentially shrinking/stretching sheet. The shrinking/stretching velocity, the free stream velocity, and the surface temperature are assumed to vary in an exponential form with the distance from the stagnation point. The skin friction coefficient and the local Nusselt number are determined for the understanding of the flow and heat transfer characteristics. The practical applications include the cooling of extruded materials in industrial processes using an inward directed fan or conical liquid jets. To the best of our knowledge, this kind of exponential shrinking/stretching sheet problem has never been considered before.

2. Problem Formulation

Consider a stagnation-point flow over an exponentially shrinking/stretching sheet immersed in an incompressible viscous fluid as shown in Figure 1. The Cartesian coordinates (x,y) are taken such that the x-axis is measured along the sheet, while the y-axis is normal to it. It is assumed that the free stream velocity, the shrinking/stretching velocity, and the surface temperature are given by $U_\infty = a\mathrm{e}^{x/L}$, $U_w = b\mathrm{e}^{x/L}$, and $T_w = T_\infty + c\,\mathrm{e}^{x/L}$, respectively, where a, b, and c are constants, and L is the reference length. The boundary layer equations are [9, 27, 28]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_\infty\frac{\mathrm{d}U_\infty}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$\begin{aligned} &u=U_w,\ \ v=0,\ \ T=T_w \quad \text{at} \quad y=0,\\ &u\to U_\infty,\ \ T\to T_\infty \quad \text{as} \quad y\to\infty, \end{aligned} \tag{4}$$

where u and v are the velocity components along the x- and y-axes, respectively, α is the thermal diffusivity of the fluid, and ν is the kinematic viscosity.

Introducing the following similarity transformation (see Magyari and Keller [9]),

$$\begin{aligned} &\eta=\left(\frac{a}{2\nu L}\right)^{1/2}\mathrm{e}^{x/(2L)}y,\ \ u=a\mathrm{e}^{x/L}f'(\eta),\\ &v=-\left(\frac{\nu a}{2L}\right)^{1/2}\mathrm{e}^{x/(2L)}(f+\eta f'),\ \ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \end{aligned} \tag{5}$$

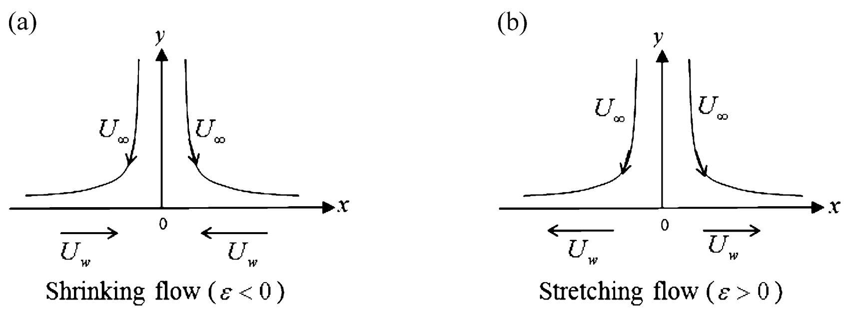

Fig. 1. Physical model and coordinate system.

the continuity equation (1) is automatically satisfied, and (2) and (3) reduce to

$$f''' + ff'' - 2f'^2 + 2 = 0, \tag{6}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + f\theta' - 2f'\theta = 0, \tag{7}$$

where primes denote differentiation with respect to η and $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number. The transformed boundary conditions are

$$\begin{aligned} &f(0) = 0, \;\; f'(0) = \varepsilon, \;\; \theta(0) = 1, \\ &f'(\eta) \to 1, \;\; \theta(\eta) \to 0 \text{ as } \eta \to \infty, \end{aligned} \tag{8}$$

with $\varepsilon = b/a$ being the shrinking/stretching parameter. We note that $\varepsilon < 0$ is valid for shrinking, $\varepsilon > 0$ for stretching, and $\varepsilon = 0$ corresponds to a fixed sheet.

The main physical quantities of interest are the skin friction coefficient and the local Nusselt number, which are proportional to the quantities $f''(0)$ and $-\theta'(0)$, respectively. Thus, our task is to investigate how the values of $f''(0)$ and $-\theta'(0)$ vary with the shrinking/stretching parameter ε and the Prandtl number Pr.

3. Numerical Method

The transformation of the governing partial differential equations into ordinary ones using similarity variables reduced the numerical work significantly. Equations (6) and (7) subject to the boundary conditions (8) are integrated numerically using a finite difference scheme known as the Keller-box method, which is described in [29, 30]. This method is unconditionally stable and has been successfully used by several authors to solve various problems in fluid mechanics and heat transfer [31–37]. The solution is obtained in the following four steps:

1. Reduce (6) and (7) to a first-order system.
2. Write the difference equations using central differences.
3. Linearize the resulting algebraic equations by Newton's method, and write them in matrix-vector form.
4. Solve the linear system by the block-tridiagonal-elimination technique.

The step size $\Delta\eta$ in η, and the position of the edge of the boundary layer η_∞ have to be adjusted for different values of parameters involved to maintain the necessary accuracy. To conserve space, the details of the solution procedure are not presented here.

4. Results and Discussion

Graphical results are presented for different physical parameters appearing in the present model. We note that (6) and (7) are decoupled, and thus the flow field is not affected by the thermal field.

Figure 2 shows the variations of the skin friction coefficient $f''(0)$ against the shrinking/stretching parameter ε, while the respective local Nusselt numbers $-\theta'(0)$ are presented in Figure 3. Two branches of solutions are found to exist within the range $\varepsilon_c < \varepsilon \le -1$, while for $\varepsilon > -1$, the solution is unique. It is seen that for negative values of ε (shrinking case), there is a critical value ε_c where the upper branch meets the lower branch. Based on our computation, $\varepsilon_c \cong -1.4872$. Beyond this critical value, no solution exists. In these figures, the solid lines denote the upper branch, while

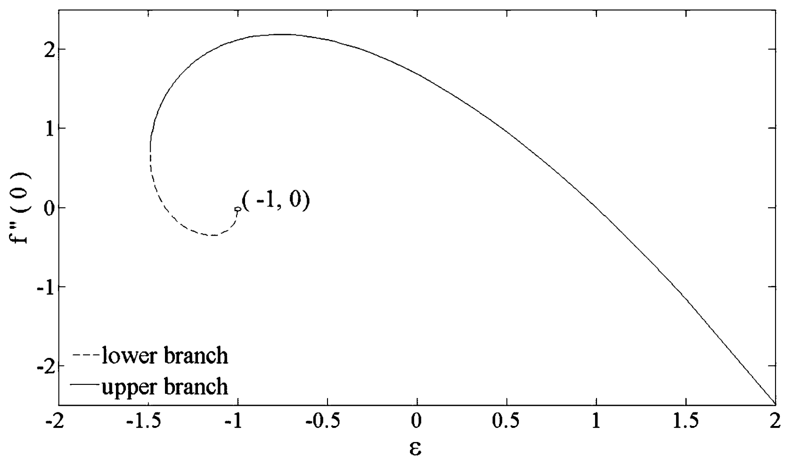

Fig. 2. Variation of the skin friction coefficient $f''(0)$ with ε.

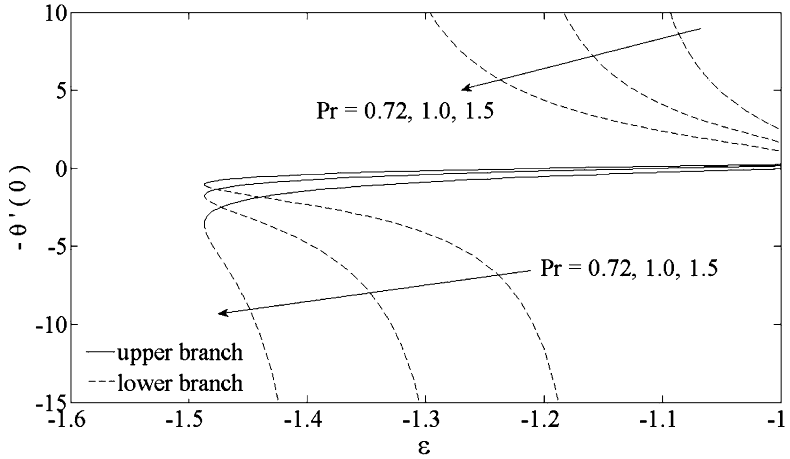

Fig. 3. Variation of the local Nusselt number $-\theta'(0)$ with ε for various values of Pr.

the dash lines denote the lower branch solutions. It is also evident from these figures, that the range of ε for which the solution exists is very small for the shrinking case. This is due to the vorticity that almost cannot be confined in the boundary layer. It is observed in Figure 3 that the lower branch solutions show discontinuity at $\varepsilon \cong -1.145$, -1.255, and -1.375 for $\mathrm{Pr} = 0.72$, 1.0, and 1.5, respectively. This phenomenon has been observed by other researchers in the literature, for different problems, for example Ridha [38] and Ishak et al. [39, 40]. Further, it is found that when $\varepsilon = 1$ (stretching case), the value of the skin friction coefficient $f''(0)$ is zero. This is because when $\varepsilon = 1$, the stretching velocity is equal to the external velocity, and thus there is no friction between the fluid and the solid surface. Furthermore, when $\varepsilon = 1$, the exact solution of (6) subject to the boundary condition (8) can be obtained, and is given by $f(\eta) = \eta$, which then implies $f''(\eta) = 0$ for all η. The present numerical result agreed with this exact solution. It is also observed that for the upper branch solution, $f''(0) > 0$ when $\varepsilon < 1$ and $f''(0) < 0$ when $\varepsilon > 1$. Physically, a positive value of $f''(0)$ means the fluid exerts a drag force on the sheet, and negative value means the opposite. On the other hand, the negative value of $f''(0)$ for the lower branch solution as shown in Figure 2 is due to the back flow, see Figure 4. The velocity gradient at the surface is negative for $\varepsilon = -1$ and $\varepsilon = -1.2$, but is positive for $\varepsilon = -1.45$, which is in agreement with the results presented in Figure 2.

Figure 5 shows the effects of $\varepsilon < 0$ (shrinking) on the temperature profiles when $\mathrm{Pr} = 1$. For the upper branch solution, with increasing negative values of ε, the temperature gradient at the surface increases, re-

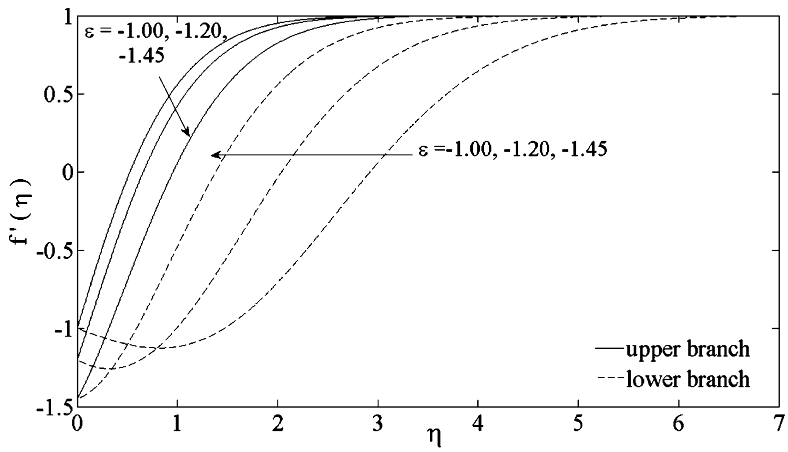

Fig. 4. Velocity profile $f'(\eta)$ for various values of $\varepsilon < 0$.

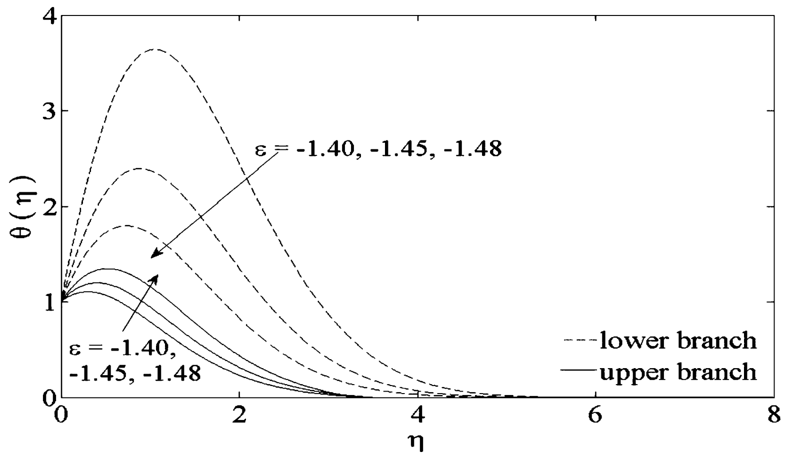

Fig. 5. Temperature profile $\theta(\eta)$ for various values of $\varepsilon < 0$ when $\mathrm{Pr} = 1$.

sulting in an increase of the local Nusselt number. The opposite trend is observed for the lower branch solution, increasing ε (in absolute sense) is to decrease the temperature gradient at the surface.

The temperature profiles for different values of Pr when $\varepsilon = -1.45$ are presented in Figure 6. It is seen that the temperature gradient at the surface increases as Pr increases. Thus, the local Nusselt number $-\theta'(0)$, which represents the heat transfer rate at the surface increases (in absolute sense) as the Prandtl number Pr increases.

The temperature overshoot shown in Figures 5 and 6 stems from the balancing act between the heat transfer from the solid boundary and its diffusion into the boundary layer and convection from the moving flows. It is dependent upon the Prandtl number Pr and the stretching/shrinking parameter ε. If the production of heat (heat transfer and diffusion) is greater than the convection term, then there will be an accumulation of heat and thus the increase of temperature. For the stretching case, there is no temperature overshoot, as shown in Figures 7 and 8. Both figures show that increasing Pr or ε decreases the thermal boundary layer thickness, and in consequence increases the local Nusselt number $-\theta'(0)$. Thus, the heat transfer rate at the surface increases as Pr or ε increases.

The velocity profiles for selected values of ε (≥ 0) are presented in Figure 9. This figure shows that the velocity gradient at the surface is zero when $\varepsilon = 1$, positive when $\varepsilon < 1$, and negative when $\varepsilon > 1$. This observation is in agreement with the results presented in Figure 2. We also note that the velocity boundary layer thickness decreases as ε increases. Finally, the velocity and temperature profiles for selected values of

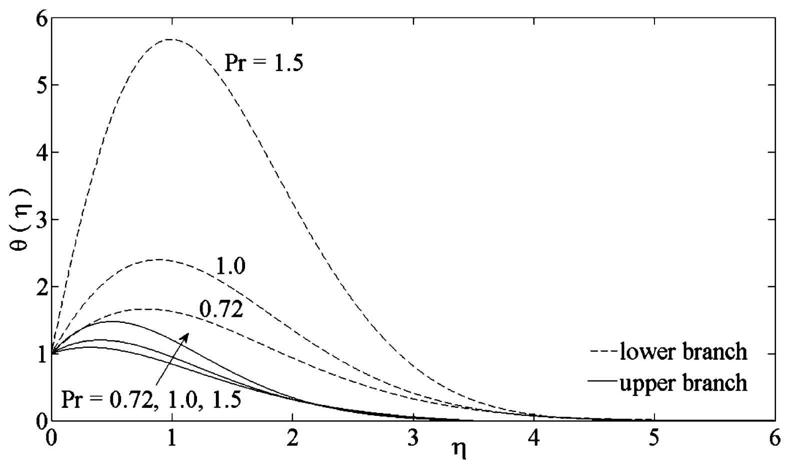

Fig. 6. Temperature profile $\theta(\eta)$ for various values of Pr when $\varepsilon = -1.45$.

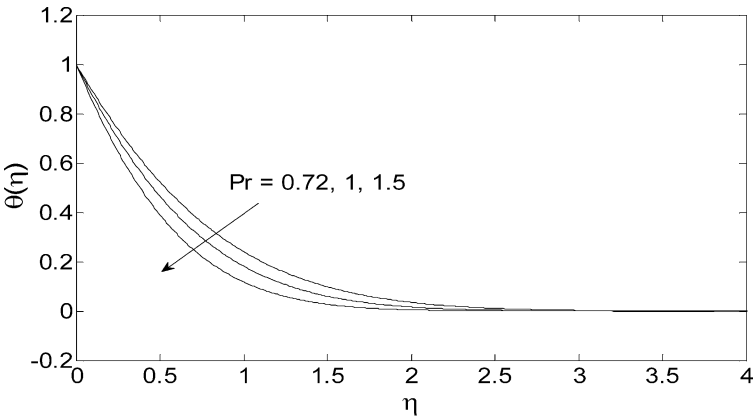

Fig. 7. Temperature profile $\theta(\eta)$ for various values of Pr when $\varepsilon = 0.5$.

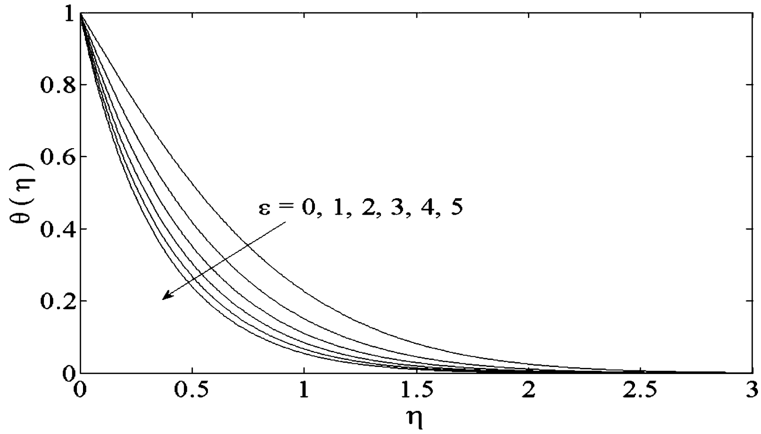

Fig. 8. Temperature profile $\theta(\eta)$ for various values of ε when $\mathrm{Pr} = 1$.

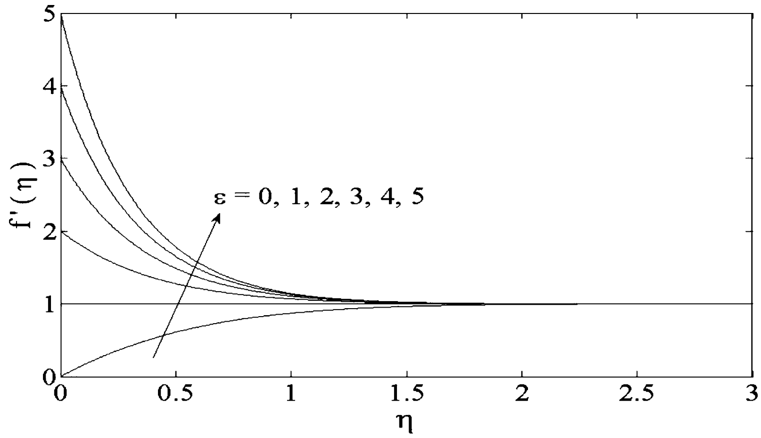

Fig. 9. Velocity profile $f'(\eta)$ for various values of ε.

parameters presented in Figures 4 – 9 show that the far field boundary conditions (8) are satisfied asymptotically, thus support the validity of the numerical results obtained, besides supporting the dual nature of the solutions to the boundary value problem (6) – (8).

5. Conclusions

The problem of stagnation-point flow over an exponentially shrinking/stretching sheet immersed in an incompressible viscous fluid was investigated numerically. Similarity solutions were obtained, and the effects of the governing parameters, namely the shrinking/stretching parameter ε and the Prandtl number Pr on the fluid flow and heat transfer characteristics were discussed. It was found that dual solutions exist for the shrinking case, while for the stretching case, the solution is unique. Moreover, it was found that increasing the Prandtl number is to increase the heat transfer rate at the surface.

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[1] M. V. Karwe and Y. Jaluria, ASME J. Heat Mass Transfer **110**, 655 (1988).
[2] M. V. Karwe and Y. Jaluria, ASME J. Heat Transfer **113**, 612 (1991).
[3] L. J. Crane, Z. Angew. Math. Phys. **21**, 645 (1970).
[4] H. I. Andersson, K. H. Back, and B. S. Dandapat, Int. J. Nonlin. Mech. **29**, 929 (1992).
[5] P. S. Gupta and A. S. Gupta, Can. J. Chem. Eng. **55**, 744 (1977).
[6] R. Nazar, N. Amin, D. Filip, and I. Pop, Int. J. Eng. Sci. **42**, 1241 (2004).
[7] M. A. Hossain and H. S. Takhar, Heat Mass Transfer **31**, 243 (1996).
[8] A. Ishak, R. Nazar, and I. Pop, Arab. J. Sci. Eng. **31**, 165 (2006).
[9] E. Magyari and B. Keller, J. Phys. D: Appl. Phys. **32**, 577 (1999).
[10] B. Bidin and R. Nazar, Euro. J. Sci. Res. **33**, 710 (2009).
[11] M. Sajid and T. Hayat, Int. Commun. Heat Mass Transfer **35**, 347 (2008).
[12] S. Nadeem, T. Hayat, M. Y. Malik, and S. A. Rajput, Z. Naturforsch. **65a**, 495 (2010).
[13] S. Nadeem, S. Zaheer, and T. Fang, Numer. Algo. **57**, 187 (2011).
[14] D. Pal, Appl. Math. Comput. **217**, 2356 (2010).
[15] A. Ishak, Sains Malaysiana **40**, 391 (2011).
[16] E. Sanjayanand and S. K. Khan, Int. J. Thermal Sci. **45**, 819 (2006).
[17] M. A. El-Aziz, Can. J. Phys. **87**, 359 (2009).
[18] M. Miklavčič and C. Y. Wang, Quart. Appl. Math. **64**, 283 (2006).
[19] T. Fang, Int. J. Heat Mass Transfer **51**, 5838 (2008).
[20] C. Y. Wang, Int. J. Nonlin. Mech. **43**, 377 (2008).
[21] M. Sajid, T. Javed, and T. Hayat, Nonlin. Dyn. **51**, 259 (2008).
[22] S. Nadeem and M. Awais, Phys. Lett. A **372**, 4965 (2008).
[23] A. Ishak, Y. Y. Lok, and I. Pop, Chem. Eng. Commun. **197**, 1417 (2010).
[24] S. Nadeem, A. Hussain, and K. Vajravelu, Z. Naturforsch. **65a**, 969 (2010).
[25] S. Nadeem, A. Hussain, and M. Khan, Z. Naturforsch. **65a**, 540 (2010).
[26] S. Nadeem, S. Abbasbandy, and M. Hussain, Z. Naturforsch. **64a**, 575 (2009).
[27] A. Ishak, K. Jafar, R. Nazar, and I. Pop, Physica A **388**, 3377 (2009).
[28] K. Bhattacharyya and G. C. Layek, Int. J. Heat Mass Transfer **54**, 302 (2011).
[29] H. B. Keller, A new difference scheme for parabolic problems, in: Numerical Solution of Partial-Differential Equations (Ed. J. Bramble), Vol. II, Academic, New York 1970, p. 327.
[30] T. Cebeci and P. Bradshaw, Physical and Computational Aspects of Convective Heat Transfer, Springer, New York 1988.
[31] A. Ishak, R. Nazar, and I. Pop, ASME J. Heat Transfer **129**, 1087 (2007).
[32] A. Ishak, R. Nazar, and I. Pop, J. Eng. Math. **62**, 23 (2008).
[33] A. Ishak, R. Nazar, and I. Pop, Meccanica **43**, 411 (2008).
[34] A. Ishak, R. Nazar, and I. Pop, Heat Mass Transfer **45**, 563 (2009).
[35] T. Javed, Z. Abbas, M. Sajid, and N. Ali, Int. J. Heat Mass Transfer **54**, 2034 (2011).
[36] F. M. Ali, R. Nazar, N. M. Arifin, and I. Pop, Int. J. Numer. Methods Fluids Doi: 10.1002/fld.2435 (2011).
[37] F. M. Ali, R. Nazar, N. M. Arifin, and I. Pop, Int. J. Numer. Methods Fluids **66**, 1465 (2011).
[38] A. Ridha, Z. Angew. Math. Phys. **47**, 341 (1996).
[39] A. Ishak, R. Nazar, N. M. Arifin, and I. Pop, Int. J. Thermal Sci. **47**, 417 (2008).
[40] A. Ishak, R. Nazar, N. Bachok, and I. Pop, Int. J. Heat Mass Transfer **53**, 4506 (2010).